Quantum scalar field in quantum gravity: the propagator and Lorentz invariance in the spherically symmetric case

Rodolfo Gambini[1], Jorge Pullin[2], Saeed Rastgoo[1]
1. Instituto de Física, Facultad de Ciencias, Iguá 4225, esq. Mataojo, Montevideo, Uruguay.
2. Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001

We recently studied gravity coupled to a scalar field in spherical symmetry using loop quantum gravity techniques. Since there are local degrees of freedom one faces the "problem of dynamics". We attack it using the "uniform discretization technique". We find the quantum state that minimizes the value of the master constraint for the case of weak fields and curvatures. The state has the form of a direct product of Gaussians for the gravitational variables times a modified Fock state for the scalar field. In this paper we do three things. First, we verify that the previous state also yields a small value of the master constraint when one polymerizes the scalar field in addition to the gravitational variables. We then study the propagators for the polymerized scalar field in flat space-time using the previously considered ground state in the low energy limit. We discuss the issue of the Lorentz invariance of the whole approach. We note that if one uses real clocks to describe the system, Lorentz invariance violations are small. We discuss the implications of these results in the light of Hořava's *Gravity at the Lifshitz point* and of the argument about potential large Lorentz violations in interacting field theories of Collins *et. al.*

This work is dedicated to Josh Goldberg for his many contributions to our understanding of space-time.

I. INTRODUCTION

In previous work [1] (hereafter referred to as Paper I), we have studied a spherically symmetric scalar field coupled to spherically symmetric gravity in the loop representation. Using symmetry adapted variables, one is left with a diffeomorphism and Hamiltonian constraint that have a non-trivial constraint algebra with structure functions, pretty much like in the full theory. We gauge fixed the diffeomorphism constraint to make things simpler but the remaining Hamiltonian constraint has an algebra with itself still involving structure functions and, up to present, no one has found a gauge fixing that would avoid this problem that leads to a Hamiltonian that is local [2]. Having structure functions poses problems for the Dirac quantization procedure. We decided to handle the situation using the "uniform discretization" approach [3]. In that approach one discretizes the theory in such a way that the evolution equations are generated by the *master constraint* (the sum of the squares of the constraints) [4]. One studies the spectrum of the quantum master constraint. If zero is in the spectrum the associated eigenstate corresponds to the continuum limit. If zero is not in the spectrum, one is left with a quantum theory that has a fundamental level of discreteness, but that can approximate the continuum theory well in many circumstances of interest. We could not conclusively prove that zero was in the kernel in the model studied. We proceeded variationally by proposing a trial state which depended on a set of parameters and minimized the value of the master constraint as a function of those parameters. The resulting state had a very small value of the master constraint for lattice spacings that are large compared to the Planck scale (but very small by, say, particle physics scales) and therefore approximates continuum general relativity in large scales very well. For simplicity we used a "polymer" representation for the gravitational variables but followed a regular quantization for the scalar field. The state has the form of a direct product of Gaussians for the gravitational variables at each lattice site times a modified Fock vacuum for the scalar field variables (the modification is due to the fact that the background is not globally flat, in 1+1 dimensions the zero point energy of the vacuum generates a deficit angle, and also that we incorporate quantum corrections to the background geometry).

In this paper we will do three things. First we will verify that the vacuum state we just discussed is a good vacuum for the polymerized theory, at least in the case in which the polymerization parameter is small. We will compute the expectation value of the master constraint for the fully polymerized theory in the vacuum state to leading order in the polymerization parameter and show that the resulting terms are very small. Second, we will study the low energy propagator for the scalar field on the above discussed quantum state. We will see that one has different options for polymerizing the scalar field and this will lead to different types of propagators. Generically they fall within the class of propagators considered by Hořava [5]. We will again work in the limit in which the polymerization parameter is small. The resulting propagators are not Lorentz invariant. We will analyze these effects in the light of the work of Collins, Pérez, Sudarsky, Urrutia and Vucetich [6] that shows that even Lorentz violations of Planck scale can have catastrophic effects when one considers interacting quantum field theories. We will see that their argument does not apply to this model if one considers real clocks to parameterize time.

It is worthwhile mentioning related recent work. Husain and Kreienbuehl [7] consider the polymerization of a scalar field without assuming spherical symmetry and proceed to define creation and annihilation operators for the polymerized theory. More recently, Hossain, Husein and Seahra [8] have analyzed the propagator in that context and have found Lorentz violations. Their work cannot be directly compared to ours for reasons we will discuss in section III B. Laddha and Varadarajan [9] consider a scalar field in $1+1$ dimensions but parameterized, including the embedding variables in their treatment. They are apparently able to recover Lorentz invariance exactly so the connection to our work is at the moment unclear.

The plan of the paper is as follows: in the next section we will discuss how justified one is in using the Fock vacuum for the scalar field in the context of a polymerized theory. In section III we discuss the polymerization of the field and the polymerization of the canonically conjugate momentum of the field and compute the resulting propagators. In section IV we address the issue of Lorentz invariance. We end the paper with a discussion.

II. APPROPRIATENESS OF USING THE FOCK VACUUM FOR THE SCALAR FIELD

In Paper I we minimized the master constraint using a variational technique that used as trial state one that consisted of Gaussians centered around flat space-time for the gravitational variables times a (curved space, quantum corrected) Fock vacuum for the scalar field. The use of the Fock vacuum appeared compelling in part due to the fact that we were not polymerizing the scalar field in our treatment. Since in this paper we will be polymerizing the scalar field, it begs the question of the appropriateness of continuing to use the Fock vacuum. In this section we would like to show that the Fock vacuum still yields a very small value for the master constraint even if one polymerizes the scalar field variables.

We start by considering the Hamiltonian of gravity coupled to a scalar field in spherical symmetry we considered in Paper I,

$$H = H_{\rm vac} + 2G\, H_{\rm matt}, \tag{1}$$

where

$$\begin{aligned} H_{\rm vac} &= \left(-x - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)' = \partial H_v(x)/\partial x, &(2)\\ H_{\rm matt} &= \frac{P_\phi^2}{2(E^\varphi)^2} + \frac{x^4(\phi')^2}{2(E^\varphi)^2} - \frac{xK_\varphi P_\phi \phi'}{E^\varphi}. &(3)\end{aligned}$$

We will now rescale the variables,

$$\begin{aligned} P_\phi^{\rm orig} &= xP_\phi^{\rm new}, &(4)\\ \phi^{\rm orig} &= \phi^{\rm new}/x, &(5)\end{aligned}$$

and will drop the “new” superscript from now on to economize in the notation. The matter Hamiltonian then becomes,

$$H_{\rm matt} = \frac{H^{(1)}}{(E^\varphi)^2} + \frac{H^{(2)}K_\varphi}{E^\varphi}, \tag{6}$$

where

$$\begin{aligned} H^{(1)} &= \frac{1}{2}P_\phi^2 x^2 + \frac{1}{2}\phi^2 - x\phi'\phi + \frac{1}{2}x^2(\phi')^2, &(7)\\ H^{(2)} &= \phi P_\phi - x\phi' P_\phi. &(8)\end{aligned}$$

We now proceed to discretize and polymerize the matter Hamiltonian,

$$H_{\rm matt}(i) = \frac{H^{(1)}(i)}{(E^\varphi(i))^2} + \frac{H^{(2)}(i)\sin\left(\rho K_\varphi(i)\right)}{\rho E^\varphi(i)}, \tag{9}$$

where,

$$\begin{aligned} H^{(1)}(i) &= \frac{\epsilon}{2}P_\phi^2(i)x(i)^2 + \frac{\epsilon^3\sin^2\left(\beta\phi(i)\right)}{2\beta^2} - \frac{\epsilon^2 x(i)}{\beta^2}\sin\left(\beta\phi(i)\right)\sin\left(\beta\left(\phi(i+1)-\phi(i)\right)\right) &(10)\\ &\quad + \frac{\epsilon x(i)^2}{2\beta^2}\sin^2\left(\beta\left(\phi(i+1)-\phi(i)\right)\right),\\ H^{(2)}(i) &= \frac{P_\phi(i)}{\beta}\left(\epsilon\sin\left(\beta\phi(i)\right) - x(i)\sin\left(\beta\left(\phi(i+1)-\phi(i)\right)\right)\right). &(11)\end{aligned}$$

We now write the complete Hamiltonian but expand the trigonometric functions in β and keep the two lowest orders, e.g. $\sin(\beta\phi)/\beta \sim \phi - \beta^2\phi^3/6$, we do this so it is clear that to leading order one will have the same results as in Paper I, and the next order will be the corrections introduced by the polymerization and we can analyze their influence. We get for (10) and (11),

$$H^{(1)}(i) = H^{(1)}_{\text{lead}}(i) + H^{(1)}_{\text{corr}}(i), \tag{12}$$

$$H^{(2)}(i) = H^{(2)}_{\text{lead}}(i) + H^{(2)}_{\text{corr}}(i), \tag{13}$$

for which "lead" refers to leading order and "corr" refers to the correction terms based on the above expansion of matter Hamiltonian in β and

$$H^{(1)}_{\text{lead}}(i) = \frac{\epsilon}{2}P_\phi(i)^2x(i)^2 + \frac{1}{2}\epsilon^3\phi(i)^2 + \frac{1}{2}\epsilon x(i)^2\left(\phi(i+1)-\phi(i)\right)^2 - \epsilon^2 x(i)\phi(i)\left(\phi(i+1)-\phi(i)\right), \tag{14}$$

$$H^{(1)}_{\text{corr}}(i) = \frac{\epsilon^3\beta^2}{6}\Bigg(-x(i)^2\frac{\left(\phi(i+1)-\phi(i)\right)^4}{\epsilon^2} + \frac{x(i)\left(\phi(i+1)-\phi(i)\right)\phi(i)^3}{\epsilon}, \tag{15}$$

$$+\frac{x(i)\left(\phi(i+1)-\phi(i)\right)^3\phi(i)}{\epsilon} - \phi(i)^4\Bigg)$$

$$H^{(2)}_{\text{lead}}(i) = \epsilon\left(-\frac{x(i)P_\phi(i)\left(\phi(i+1)-\phi(i)\right)}{\epsilon} + P_\phi(i)\phi(i)\right), \tag{16}$$

$$H^{(2)}_{\text{corr}}(i) = \frac{\epsilon\beta^2}{6}\left(\frac{x(i)P_\phi(i)\left(\phi(i+1)-\phi(i)\right)^3}{\epsilon} - P_\phi(i)\phi(i)^3\right). \tag{17}$$

These should be put into (9) to give the matter Hamiltonian expanded in β. We are now going to focus on the master constraint. It can be written as,

$$\mathbb{H}(i) = c_{11}(i)\left(H^{(1)}(i)\right)^2 + c_1(i)H^{(1)}(i) + c_{12}(i)H^{(1)}(i)H^{(2)}(i) + c_{22}(i)\left(H^{(2)}(i)\right)^2 + c_2(i)H^{(2)}(i). \tag{18}$$

Where the c coefficients depend only on the gravitational variables. Substituting the leading order terms of (14) and (16) into (18), yields the results of paper I (taking into account the re-scalings (4) and (5)). What we want to show now is that substituting the correction terms into the master constraint (18) and taking its expectation value with respect to the trial vacuum state of the previous paper, yields corrective terms which are very small.

For this, we observe that the contribution of the correction terms to the master constraint can be written as,

$$\mathbb{H}_{\text{corr}}(i) = c_{11}(i)\left(H^{(1)}_{\text{lead}}(i)H^{(1)}_{\text{corr}}(i)\right) + c_1(i)H^{(1)}_{\text{corr}}(i) + c_{12}(i)\left(H^{(1)}_{\text{lead}}(i)H^{(2)}_{\text{corr}}(i) + H^{(2)}_{\text{lead}}(i)H^{(1)}_{\text{corr}}(i)\right) \tag{19}$$

$$+c_{22}(i)\left(H^{(2)}_{\text{lead}}(i)H^{(2)}_{\text{corr}}(i)\right) + c_2(i)H^{(2)}_{\text{corr}}(i) + c_{00}(i)$$

and we want to show that $\langle\psi^{\text{trial}}_{\vec{\sigma}}|\mathbb{H}_{\text{corr}}(i)|\psi^{\text{trial}}_{\vec{\sigma}}\rangle$ is very small.

Our strategy is the following: We compute the dominant terms by first going to the continuum limit and writing (14)-(17) in their continuum limit form by using,

$$P_\phi(i) = \epsilon P_\phi(x), \tag{20}$$

$$E^\phi(i) = \epsilon E^\phi(x), \tag{21}$$

$$\phi(i) = \phi(x), \tag{22}$$

$$\frac{\phi(i+1)-\phi(i)}{\epsilon} = \frac{\partial\phi(x)}{\partial x}. \tag{23}$$

We then substitute in the result the continuum form of (14)-(17) that we just calculated and also the Fourier expansions of the $\phi(x)$ and its conjugate momentum $P_\phi(x)$ fields, which are,

$$\phi(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega\frac{\left(C(\omega)e^{-i\omega t} + \bar{C}(\omega)e^{i\omega t}\right)\sin(\omega x)}{\sqrt{\pi\omega}}, \tag{24}$$

and

$$P_\phi(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega\frac{-i\omega\left(C(\omega)e^{-i\omega t} - \bar{C}(\omega)e^{i\omega t}\right)\sin(\omega x)}{\sqrt{\pi\omega}}. \tag{25}$$

Next, using the expanded version of the terms (14)-(17) resulting from the substitution of the Fourier expansion of the fields, we find the individual terms that constitute (19), meaning the terms that appear multiplied by c_m and c_{kl}'s in (19) (i.e. $H^{(1)}_{\text{lead}}(x)H^{(1)}_{\text{corr}}(x)$ etc).

From here on, let us focus on one of the individual terms that make up (19) (an arbitrary one). We can then repeat the process for all the other terms. We proceed to find the portions of the individual terms that do not have vanishing expectation values, taking into account that $C(\omega)$ and $\bar{C}(\omega)$ are annihilation and creation operators. It turns out that we encounter terms with four $C(\omega)$ and/or $\bar{C}(\omega)$ operators for non-cross terms like for example $H^{(1)}_{\text{corr}}(x)$ and six of them in cross terms like $H^{(1)}_{\text{lead}}(x)H^{(1)}_{\text{corr}}(x)$. Then the parts of the non-cross terms with four operators with non-vanishing expectation values just include the terms with

$$C_4\bar{C}_3C_2\bar{C}_1, \tag{26}$$
$$C_4C_3\bar{C}_2\bar{C}_1, \tag{27}$$

where we wrote $C(\omega_1)$ as C_1, etc. for the sake of brevity. Also the parts of the cross terms with six operators with non-vanishing expectation values turn out to include only the terms with

$$C_6C_5C_4\bar{C}_3\bar{C}_2\bar{C}_1, \tag{28}$$
$$C_6C_5\bar{C}_4C_3\bar{C}_2\bar{C}_1, \tag{29}$$
$$C_6\bar{C}_5C_4C_3\bar{C}_2\bar{C}_1, \tag{30}$$
$$C_6C_5\bar{C}_4\bar{C}_3C_2\bar{C}_1, \tag{31}$$
$$C_6\bar{C}_5C_4\bar{C}_3C_2\bar{C}_1. \tag{32}$$

Using the commutation relation

$$\left[\hat{C}(\omega_1),\hat{\bar{C}}(\omega_2)\right] = \delta(\omega_1-\omega_2), \tag{33}$$

we evaluate the expectation values of the relevant parts (of the individual term) we are working with. We will have

$$\begin{aligned}
\langle C_4\bar{C}_3C_2\bar{C}_1\rangle &= \delta(\omega_4-\omega_3)\delta(\omega_2-\omega_1), && (34)\\
\langle C_4C_3\bar{C}_2\bar{C}_1\rangle &= \delta(\omega_4-\omega_2)\delta(\omega_3-\omega_1)+\delta(\omega_4-\omega_1)\delta(\omega_3-\omega_2), && (35)\\
\langle C_6C_5C_4\bar{C}_3\bar{C}_2\bar{C}_1\rangle &= \delta(\omega_6-\omega_3)[\delta(\omega_5-\omega_2)\delta(\omega_4-\omega_1)+\delta(\omega_5-\omega_1)\delta(\omega_4-\omega_2)] && (36)\\
&\quad+\delta(\omega_6-\omega_2)[\delta(\omega_5-\omega_3)\delta(\omega_4-\omega_1)+\delta(\omega_5-\omega_1)\delta(\omega_4-\omega_3)]\\
&\quad+\delta(\omega_6-\omega_1)[\delta(\omega_5-\omega_3)\delta(\omega_4-\omega_2)+\delta(\omega_5-\omega_2)\delta(\omega_4-\omega_3)],\\
\langle C_6C_5\bar{C}_4C_3\bar{C}_2\bar{C}_1\rangle &= \delta(\omega_6-\omega_4)[\delta(\omega_5-\omega_2)\delta(\omega_3-\omega_1)+\delta(\omega_5-\omega_1)\delta(\omega_3-\omega_2)] && (37)\\
&\quad+\delta(\omega_5-\omega_4)[\delta(\omega_6-\omega_2)\delta(\omega_3-\omega_1)+\delta(\omega_6-\omega_1)\delta(\omega_3-\omega_2)],\\
\langle C_6\bar{C}_5C_4C_3\bar{C}_2\bar{C}_1\rangle &= \delta(\omega_6-\omega_5)[\delta(\omega_4-\omega_2)\delta(\omega_3-\omega_1)+\delta(\omega_4-\omega_1)\delta(\omega_3-\omega_2)], && (38)\\
\langle C_6C_5\bar{C}_4\bar{C}_3C_2\bar{C}_1\rangle &= \delta(\omega_2-\omega_1)[\delta(\omega_6-\omega_4)\delta(\omega_5-\omega_3)+\delta(\omega_6-\omega_3)\delta(\omega_5-\omega_4)], && (39)\\
\langle C_6\bar{C}_5C_4\bar{C}_3C_2\bar{C}_1\rangle &= \delta(\omega_6-\omega_5)\delta(\omega_4-\omega_3)\delta(\omega_2-\omega_1). && (40)
\end{aligned}$$

Finally, we add up the expectation values of the relevant parts resulting from the previous step (these results are the non-vanishing expectation-values parts of the individual term), to get the complete expectation value of the individual term we chose. We now repeat the procedure to get the complete expectation value of all the other individual terms that build up (19) and after that, add up all the results to get the expectation value of $\mathbb{H}_{\text{corr}}(x)$.

Then we convert the resulting expectation value $\langle\mathbb{H}_{\text{corr}}(x)\rangle$ back to its discrete form, $\langle\mathbb{H}_{\text{corr}}(i)\rangle$, by reversing the continuum limit and neglecting highly oscillating terms like $\sin(\frac{n\pi x}{\epsilon})$ and the similar cosine and Ci terms. Expanding the result in ℓ_p, collecting the terms of the order of β^2, and expanding it in ϵ, we find that the leading term of corrections are of order

$$\langle\mathbb{H}_{\text{corr}}(x)\rangle \sim \frac{\ell_p^5\ln\left(\frac{\pi x}{\epsilon}\right)^2}{\epsilon\pi x^4}\beta^2. \tag{41}$$

This leading term is actually the expectation value of a master constraint density. In order to get the expectation value of the master constraint itself, we need to integrate the above term with respect to x which will yield relevant terms of order

$$\int_\epsilon^L \langle\mathbb{H}_{\text{corr}}(x)\rangle dx \sim \frac{\ell_p^5}{\epsilon^4}\beta^2. \tag{42}$$

But in the previous paper, for the master constraint density, we had the leading order result of the form

$$\langle \mathbb{H}_{\text{lead}}(x) \rangle \sim \frac{\ell_p^3}{\epsilon x^2}, \tag{43}$$

where here "lead" means not the leading term of the corrections but the leading term of the expectation value of the master constraint density. Thus integrating with respect to x as above will give us the master constraint relevant terms of the order

$$\int_\epsilon^L \langle \mathbb{H}_{\text{lead}}(x) \rangle dx \sim \frac{\ell_p^3}{\epsilon^2}. \tag{44}$$

Thus we see that the corrections to the master constraint are indeed considerably smaller than the leading contributions, provided that the lattice spacing ϵ is large compared to the Planck length (but still small compared to particle physics scales, as we discussed in more detail in paper I).

III. LOW ENERGY PROPAGATORS FOR THE SCALAR FIELD

A. The standard treatment

In the previous section we have shown that the vacuum of the theory is well approximated by the tensor product state obtained variationally in Paper I. For such a state the space-time metric is locally flat with a global deficit angle. We would like to study the propagator of the polymerized scalar field in such a metric and determine possible corrections to the usual propagator introduced by the polymerization. We will study the propagator perturbatively in β, the polymerization coefficient, assuming the latter is small.

The Hamiltonian for a scalar field in spherical symmetry on a locally flat background is given by,

$$H = \frac{P_\phi^2}{2x^2} + \frac{x^2 \left(\phi'\right)^2}{2} \tag{45}$$

where x is the radial coordinate. In order to study the modes of the resulting equation of motion it is convenient to introduce a rescaling $\tilde{P}_\phi = P_\phi/x$ and $\tilde{\phi} = x\phi$. We drop the tildes from now on to simplify the notation. The Hamiltonian then becomes (ignoring boundary terms),

$$H = \frac{P_\phi^2}{2} + \frac{\left(\phi'\right)^2}{2}. \tag{46}$$

The resulting wave equation can be solved in Fourier space,

$$\phi(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} dk \frac{\left(C(\omega(k))e^{-i\omega(k)t} + \bar{C}(\omega(k))e^{i\omega(k)t}\right)\sin\left(|k|x\right)}{\sqrt{\pi\omega}}, \tag{47}$$

and in this case the dispersion relation is very simple $\omega = |k|$ and with this one can easily reconstruct the solution to the original form of the wave equation before the rescaling.

Let us now consider the discretized version of the Hamiltonian,

$$H(i) = \frac{P_\phi(i)^2}{2\epsilon} + \frac{\left(\phi(i+1) - \phi(i)\right)^2}{2\epsilon}, \tag{48}$$

where the ϵ in the first term is a remnant of the fact that the momentum is a density. The resulting discrete wave equation can be solved in modes,

$$\phi(j) = \sum_{n=-N}^{N} \frac{1}{\sqrt{2N\omega(n)}} \left(C(\omega(n))e^{-i\omega(n)t} + \bar{C}(\omega(n))e^{i\omega(n)t}\right) \text{sgn}(n) \sin\left(\frac{j\pi n}{N}\right), \tag{49}$$

where all the sums from $-N$ to N exclude zero since there is a minimum value for the momentum in a box. The frequencies are given by $\omega(n) = |2\sin(\pi n/(2N))/\epsilon|$. For further computations it is useful to define $p\,(n) \equiv \pi n/L$ and $L = N\epsilon$ and

$$\phi(n,t) \equiv \frac{1}{\sqrt{\omega(n)}} \left(C(\omega(n))e^{-i\omega(n)t} + \bar{C}(\omega(n))e^{i\omega(n)t}\right) \text{sgn}(n). \tag{50}$$

The momentum is given by,

$$P_\phi(j) = \sum_{n=-N}^{N} \frac{i}{\sqrt{2N\omega(n)}} \left(-\omega(n)C(\omega(n))e^{-i\omega(n)t} + \omega(n)\bar{C}(\omega(n))e^{i\omega(n)t}\right) \mathrm{sgn}(n) \sin\left(\frac{j\pi n}{N}\right) \epsilon, \tag{51}$$

and we define,

$$P_\phi(n,t) = \frac{i}{\sqrt{\omega(n)}} \left(-\omega(n)C(\omega(n))e^{-i\omega(n)t} + \omega(n)\bar{C}(\omega(n))e^{i\omega(n)t}\right) \mathrm{sgn}(n)\epsilon, \tag{52}$$

One can quantize the fields, with discrete commutation relations,

$$\left[\hat{\phi}(i), \hat{P}_\phi(j)\right] = i\delta_{i,j}, \tag{53}$$

which naturally lead to the introduction of the creation and annihilation operators,

$$\left[\hat{C}(\omega(n)), \hat{\bar{C}}(\omega(m))\right] = \frac{1}{2\epsilon}(\delta_{n,m} + \delta_{n,-m}). \tag{54}$$

With this one can compute the free propagators. The Feynman propagator is given by,

$$G^{(0)}(n,t,n',t') = \langle 0|T\left(\phi(n,t), \phi(n',t')\right)|0\rangle = D(n,t,t')\left(\delta_{n,n'} - \delta_{-n,n'}\right), \tag{55}$$

where T is the time ordered product and

$$D(n,t,t') = \left[\frac{\Theta(t-t')\exp\left(-i\omega(n)(t-t')\right)}{\epsilon\omega(n)} + \frac{\Theta(t'-t)\exp\left(-i\omega(n)(t'-t)\right)}{\epsilon\omega(n)}\right], \tag{56}$$

or, using the residue theorem,

$$D(n,t,t') = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\epsilon} \frac{1}{\omega^2 - \omega(n)^2 + i\sigma} \exp\left(-i\omega(t'-t)\right) \tag{57}$$

The previous expressions were in Fourier space. In direct space, one has,

$$G^{(0)}(j,t,k,t') = \sum_{n=-N}^{N} \sum_{n'=-N}^{N} \frac{1}{N} \sin\left(\frac{j\pi n}{N}\right) \sin\left(\frac{k\pi n'}{N}\right) G^{(0)}(n,t,n',t'). \tag{58}$$

B. Polymerizing the scalar field

Having computed the free propagator we now turn to study the polymerized propagator. We start by noticing that the Hamiltonian can be rewritten (again ignoring boundary terms) as,

$$H = \sum_i H(i) = \sum_i \frac{P_\phi(i)^2}{2\epsilon} + \frac{\left(\phi(i+1) - \phi(i)\right)^2}{2\epsilon} = \sum_i \frac{P_\phi(i)^2}{2\epsilon} - \frac{\left(\phi(i+1) + \phi(i-1) - 2\phi(i)\right)\phi(i)}{2\epsilon}, \tag{59}$$

and the rearrangement makes the expression appear more readily symmetric in $i+1$ and $i-1$. We proceed to polymerize,

$$H = \sum_i \left(\frac{P_\phi(i)^2}{2\epsilon} - \frac{\sin\left(\beta\left(\phi(i+1) + \phi(i-1) - 2\phi(i)\right)\right)\sin(\beta\phi(i))}{2\epsilon\beta^2}\right). \tag{60}$$

At this point some comments are in order. There are many possible choices at the time of polymerizing the theory. For instance, we could have chosen to polymerize $\phi(i+1) + \phi(i-1) - 2\phi(i)$ as we did or we could have polymerized each term in the sum individually. In the lattice one can also choose to polymerize the momentum $P_\phi(i)$ (in the continuum this may be more difficult since P is a density)[1]. If one polymerizes $P_\phi(i)$, when one takes the continuum

[1] This is the reason our work is not easily compared with that of Hossain, Husain and Seahra [8]. They polymerize the momentum in the continuum. The density nature of the momentum leads them to a polymerization parameter that is dimensionful, unlike our case.

limit, since the continuum momentum is $P_\phi(i)/\epsilon$, one would get for the first term in the Hamiltonian $\sin^2(\beta\epsilon P_\phi)/\beta^2\epsilon$ and in the limit $\epsilon \to 0$ one would recover a non-polymerized theory and therefore we would not be making contact with usual loop quantum gravity results. Polymerizing the fields as we have chosen yields in the continuum limit a term $\phi''(x)\sin(\beta\phi(x))/\beta$ showing that the continuum theory is polymerized. It is interesting to notice that spatial derivatives of fields are well defined in the Bohr compactification even if the field operators themselves are not. In this section we will work with a polymerization of the field rather than of the momentum. In a discrete theory polymerizing either fields or momenta is possible, but it does not lead to equivalent theories. For completeness, in the next section we will discuss the theory that results from polymerizing the momenta. In previous treatments in the continuum [10] the scalar field has been polymerized, although in the case of the harmonic oscillator, which one can consider closely related to a scalar field, a polymerization of the momentum has been preferred [11].

We are going to work perturbatively, expanding in β. The Hamiltonian we will consider is $H = H_0 + H_{\rm int}$ with

$$H_0 = \sum_i \left(\frac{P_\phi(i)^2}{2\epsilon} - \frac{\phi(i)\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)}{2\epsilon} \right), \tag{61}$$

and

$$H_{\rm int} = \sum_i \frac{1}{2\epsilon}\left(\frac{1}{6}\phi(i)\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)^3 + \frac{1}{6}\phi(i)^3\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)\right)\beta^2. \tag{62}$$

This interaction Hamiltonian comes from expansion in beta and keeping the first two leading terms. With it we compute the interacting propagator to leading order,

$$G^{(2)}(j,t,k,t') = G^{(0)}(j,t,k,t') + \frac{i^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \sum_{j'=-N}^{N}\sum_{k'=-N}^{N} \langle 0| T\left(\phi(j,t)\phi(k,t')H_{\rm int}(j',t_1)H_{\rm int}(k',t_2)\right)|0\rangle \tag{63}$$

To compute this expression it is convenient to rewrite the interaction Hamiltonian in momentum space (we use letters up to k for the field representation and letters starting with m for the momentum representation)

$$\begin{aligned} H_{\rm int}(j',t_1) \;=\; & \sum_{n,m,p,q=-N}^{N} \left\{ \frac{1}{48N^2}\beta^2\epsilon^5 \sin\left(\frac{\pi j' n}{N}\right)\phi(n,t_1)\omega(m)^2 \sin\left(\frac{\pi j' m}{N}\right)\phi(m,t_1) \right. \\ & \times \omega(p)^2 \sin\left(\frac{\pi j' p}{N}\right)\phi(p,t_1)\omega(q)^2\sin\left(\frac{\pi j' q}{N}\right)\phi(q,t_1) \\ & + \frac{1}{48N^2}\beta^2\epsilon \sin\left(\frac{\pi j' n}{N}\right)\phi(n,t_1)\sin\left(\frac{\pi j' m}{N}\right)\phi(m,t_1) \\ & \left. \times \sin\left(\frac{\pi j' p}{N}\right)\phi(p,t_1)\omega(q)^2\sin\left(\frac{\pi j' q}{N}\right)\phi(q,t)\right\}. \end{aligned} \tag{64}$$

We now use the identity,

$$\begin{aligned} \Delta(n,m,p,q) \;\equiv\; & \sum_{j'=-N}^{N} \frac{4}{N^2}\sin\left(\frac{\pi j' n}{N}\right)\sin\left(\frac{\pi j' m}{N}\right)\sin\left(\frac{\pi j' p}{N}\right)\sin\left(\frac{\pi j' q}{N}\right) \\ =\; & \frac{1}{N}\left[\delta_{n+m,p+q}+\delta_{n+p,m+q}+\delta_{n+q,m+p}+\delta_{n+m+p+q}-\delta_{n,m+p+q}-\delta_{m,n+p+q}-\delta_{p,n+m+q}-\delta_{q,n+m+p}\right]. \end{aligned} \tag{65}$$

We can use this identity to get,

$$\sum_{j'=-N}^{N} H_{\rm int}(j',t_1) = \frac{1}{192}\sum_{n,m,p,q=-N}^{N}\phi(n,t_1)\phi(m,t_1)\phi(p,t_1)\phi(q,t_1)\left[\left(\omega(m)^2\omega(p)^2\epsilon^4+1\right)\epsilon\,\omega(q)^2\right]\beta^2\Delta(n,m,p,q), \tag{66}$$

where we will use the notation,

$$f(m,p,q) = \left[\left(\omega(m)^2\omega(p)^2\epsilon^4+1\right)\epsilon\,\omega(q)^2\right]. \tag{67}$$

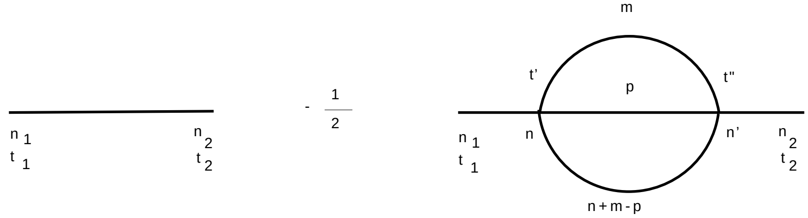
Putting everything together we get,

$$
\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) \;=\;& G^{(0)}(n_1,t_1,n_2,t_2)+\frac{i^2}{2!}\langle 0|T\big(\phi(n_1,t_1)\phi(n_2,t_2) \\
&\times\frac{1}{192}\int_{-\infty}^{\infty}dt'\sum_{n,m,p,q=-N}^{N}:\phi(n,t')\phi(m,t')\phi(p,t')\phi(q,t'):f(n,m,p)\beta^2\Delta(n,m,p,q) \\
&\times\frac{1}{192}\int_{-\infty}^{\infty}dt''\sum_{n',m',p',q'=-N}^{N}:\phi(n',t'')\phi(m',t'')\phi(p',t'')\phi(q',t''):f(n',m',p')\beta^2\Delta(n',m',p',q')\big)|0\rangle.
\end{aligned} \tag{68}
$$

Using Wick's theorem the above expression can be rewritten as a sum of diagrams of the form,

$$
\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) \;=\;& G^{(0)}(n_1,t_1,n_2,t_2)-\frac{32}{3N^2}\sum_{m,p=-N}^{N}\int_{-\infty}^{\infty}dt'dt''\left[D(n_1,t_1,t')D(m,t',t'')\right. \\
&\times\left. D(p,t',t'')D(n+m-p,t',t'')D(n_2,t'',t_2)\right]f^2(m,p,n+m-p)\beta^4\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right),
\end{aligned} \tag{69}
$$

or, graphically,



It is now convenient to Fourier transform in time,

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) \;=\;& \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-\omega(n_1)^2+i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \\
&-\frac{32}{3}\frac{1}{2}\frac{4\pi i}{\epsilon\left(\omega_1^2-\omega(n_1)^2+i\sigma\right)}\sum_{m,p=-N}^{N}\int_{-\infty}^{\infty}d\omega'd\omega'' \\
&\times\frac{4\pi i}{\epsilon\left((\omega')^2-\omega(m)^2+i\sigma\right)}\frac{4\pi i}{\epsilon\left((\omega'')^2-\omega(p)^2+i\sigma\right)} \\
&\times\frac{4\pi i}{\left(\left(\omega_1-\omega'-\omega''\right)^2-\omega(n_1-m-p)^2+i\sigma\right)}f^2(m,p,n_1-m-p)\beta^4 \\
&\times\frac{4\pi i}{\left(\omega_2^2-\omega(n_2)^2+i\sigma\right)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right),
\end{aligned} \tag{70}
$$

and the sums can be converted to integrals. Care should be exercised not to allow the denominators to vanish, since in the original discrete expression the denominators did not vanish. We recall that $\omega(n)=|2\sin(\pi n/(2N))/\epsilon|$ and $p(n)\equiv \pi n/L$ with $L=N\epsilon$. Therefore $\omega(n)=2|\sin(\epsilon p(n)/2)|/\epsilon\sim p(n)$. One then approximates,

$$
\sum_{m=1}^{N}\rightarrow\frac{L}{\pi}\int_{\pi/L}^{\pi/\epsilon}dp \tag{71}
$$

and the sum from $-N$ to 1 takes an analogous form. The expression for the Green function up to second order is,

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) \;=\;& \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2+i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \\
&-\frac{32}{3}\frac{1}{2}\frac{4\pi i}{\epsilon\left(\omega_1^2-p(n_1)^2+i\sigma\right)}\frac{1}{\pi^2}\int_{-\infty}^{\infty}d\omega'd\omega''\left[\int_{-\pi/\epsilon}^{-\pi/L}+\int_{\pi/L}^{\pi/\epsilon}\right]dp_1dp_2 \\
&\times\frac{4\pi i}{\epsilon\left((\omega')^2-{p_1}^2+i\sigma\right)}\frac{4\pi i}{\epsilon\left((\omega'')^2-{p_2}^2+i\sigma\right)}
\end{aligned} \tag{72}
$$

$$\times \frac{4\pi i}{\left((\omega_1-\omega'-\omega'')^2 - p(n_1-p_1-p_2)^2 + i\sigma\right)} \tilde{f}^2(p_1,p_2,p(n_1)-p_1-p_2)\beta^4$$
$$\times \frac{4\pi i}{(\omega_2^2-\omega(n_2)^2+i\sigma)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)$$

where

$$\tilde{f}(p_1,p_2,p(n_1)+p_1-p_2) = \left(\epsilon^4\left({p_1}^2{p_2}^2\right)+1\right)\left(p(n_1)+p_1-p_2\right)^2\epsilon^2 \tag{73}$$

The integrals can be computed by an analytic extension to the Euclidean theory and by carrying out an expansion in $p\epsilon$, the next expression is correct up to order $O(\epsilon^4 p^4)$. That is, we are assuming the wavelength of the scalar field is much larger than the lattice spacing. If one takes into account powers higher than $p\epsilon$ one has higher corrections in powers of p. The result, not including those terms, is,

$$G^{(2)}(n_1,\omega_1,n_2,\omega_2) = G^{(0)}(n_1,\omega_1,n_2,\omega_2) + \left[\frac{\alpha_1\beta^4}{\epsilon^2}+\beta^4\alpha_2 p(n_1)^2\right]\frac{4\pi i}{\epsilon}\frac{\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)}{\left(\omega_1^2-p(n_1)^2+i\sigma\right)^2} \tag{74}$$

$$= \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2\left(1+\alpha_2\beta^4\right)-\frac{\alpha_1\beta^4}{\epsilon^2}+i\sigma}\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\delta(\omega_1-\omega_2) \tag{75}$$

where α_1 and α_2 are constants of order one, and the last expression yields (74) if one expands assuming β^4 is small.

C. Polymerizing the momentum of the field

We now discuss the choice of polymerizing the momentum. As before, we write the Hamiltonian as,

$$H = \sum_i \frac{P_\phi(i)^2}{2\epsilon} - \frac{\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)\phi(i)}{2\epsilon}, \tag{76}$$

We proceed to polymerize,

$$H = \sum_i \frac{\sin^2\left(\beta P_\phi(i)\right)}{2\beta^2\epsilon} - \frac{\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)\phi(i)}{2\epsilon}, \tag{77}$$

As before, we work perturbatively, expanding in β. The Hamiltonian we will consider is $H = H_0 + H_{\rm int}$ with

$$H_{\rm int}(i) = -\frac{1}{6\epsilon}\beta^2 P_\phi(i)^4. \tag{78}$$

And we can now write the Green function up to second order,

$$G^{(2)}(j,t,k,t') = G^{(0)}(j,t,k,t') + \frac{i^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\sum_{j'=-N}^{N}\sum_{k'=-N}^{N}\langle 0|T\left(\phi(j,t)\phi(k,t')H_{\rm int}(j',t_1)H_{\rm int}(k',t_2)\right)|0\rangle \tag{79}$$

We now rewrite,

$$\sum_{j'=-N}^{N} H_{\rm int}(j',t_1) = -\frac{1}{96\epsilon}\sum_{n,m,p,q=-N}^{N} P_\phi(n,t')P_\phi(m,t')P_\phi(p,t')P_\phi(q,t')\Delta(n,m,p,q)\beta^2 \tag{80}$$

Putting everything together we get,

$$G^{(2)}(n_1,t_1,n_2,t_2) = G^{(0)}(n_1,t_1,n_2,t_2) + \frac{i^2}{2!}\langle 0|T\big(\phi(n_1,t_1)\phi(n_2,t_2) \tag{81}$$
$$\times\frac{1}{96}\int_{-\infty}^{\infty}\frac{dt'}{\epsilon}\sum_{n,m,p,q=-N}^{N} : P_\phi(n,t')P_\phi(m,t')P_\phi(p,t')P_\phi(q,t') : \beta^2\Delta(n,m,p,q)$$
$$\times\frac{1}{96}\int_{-\infty}^{\infty}\frac{dt''}{\epsilon}\sum_{n',m',p',q'=-N}^{N} : P_\phi(n',t'')P_\phi(m',t'')P_\phi(p',t'')P_\phi(q',t'') : \beta^2\Delta(n',m',p',q')\big)|0\rangle.$$

If we now use Wick's theorem as we did before, there will appear contractions not only of ϕ with itself, but also between ϕ and its momentum. Taking into account that the momentum is related to the derivative of the field $P_\phi = \epsilon\dot{\phi}$ one can compute the expectation values of products of the field and momentum or products of the momenta by taking derivatives of (57) with respect to time.

$$\begin{aligned} G^{(2)}(n_1,t_1,n_2,t_2) &= G^{(0)}(n_1,t_1,n_2,t_2) - 128\frac{\beta^4}{3N^2\epsilon^2}\sum_{m,p,q,m',p',q'=-N}^{N}\int_{-\infty}^{\infty} dt'dt'' D_{\phi P_\phi}(n_1,t_1,m,t') \\ &\quad \times D_{P_\phi P_\phi}(p,t',p',t'')D_{P_\phi P_\phi}(q,t',q',t'')D_{P_\phi P_\phi}(m+p-q,t',m'+p'-q',t'')D_{P_\phi \phi}(m',t'',n_2,t_2) \end{aligned} \tag{82}$$

where

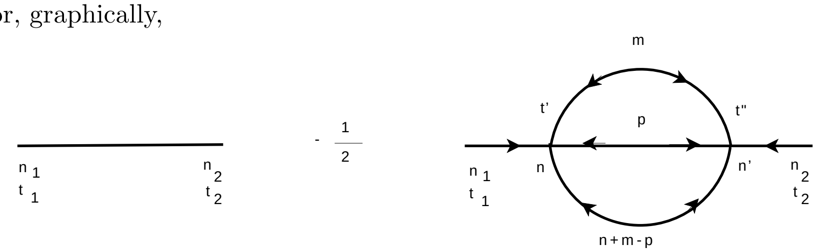
$$D_{\phi\phi}(n_1,t_1,n_2,t_2) = \frac{iL^2}{\pi\epsilon}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \tag{83}$$

$$D_{P_\phi\phi}(n_1,t_1,n_2,t_2) = -\frac{L^2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)\,d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \tag{84}$$

$$D_{\phi P_\phi}(n_1,t_1,n_2,t_2) = \frac{L^2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)\,d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \tag{85}$$

$$D_{P_\phi P_\phi}(n_1,t_1,n_2,t_2) = -\frac{iL^2\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)^2\,d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \tag{86}$$

or, graphically,



where the direction of the arrows depend on the order of appearance of ϕ and P_ϕ in their product, meaning an arrow to the right is ϕP_ϕ, an arrow to the left $P_\phi\phi$, two arrows mean $P_\phi P_\phi$, and no arrow means $\phi\phi$.

We Fourier transform in time and take the continuum approximation for the sums in p and q,

$$\begin{aligned} G^{(2)}(n_1,\omega_1,n_2,\omega_2) &= \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2+i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)+ \\ &\quad \frac{128}{3}\frac{1}{2\pi^2}\frac{1}{(\omega_1^2-p(n_1)^2+i\sigma)}\frac{1}{\pi^2}\int_{-\infty}^{\infty}d\omega'd\omega''\left[\int_{-\pi/\epsilon}^{-\pi/L}+\int_{\pi/L}^{\pi/\epsilon}\right]dp_1dp_2\left(\frac{i}{2\pi^3}\right)^3 \\ &\quad \times\frac{\epsilon p_1^2}{((\omega')^2-{p_1}^2+i\sigma)}\frac{\epsilon p_2^2}{((\omega'')^2-{p_2}^2+i\sigma)} \\ &\quad \times\frac{\left(p(n_1)-p_1-p_2\right)^2}{\left((\omega_1-\omega'-\omega'')^2-(p(n_1)-p_1-p_2)^2+i\sigma\right)}\beta^4 \\ &\quad \times\frac{4\pi i\omega(n_1)^2}{(\omega_2^2-\omega(n_2)^2+i\sigma)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \end{aligned} \tag{87}$$

The integrals can be computed as before, expanding in $p\,\epsilon$ and analytically continuing to the Euclidean theory,

$$G^{(2)}(n_1,\omega_1,n_2,\omega_2) = G^{(0)}(n_1,\omega_1,n_2,\omega_2)+\beta^4\alpha_2 p(n_1)^2\frac{4\pi i}{\epsilon}\frac{\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)}{\left(\omega_1^2-p(n_1)^2+i\sigma\right)^2} \tag{88}$$

$$= \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2\left(1+\alpha_2\beta^4\right)+i\sigma}\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\delta(\omega_1-\omega_2) \tag{89}$$

where the last expression yields (88) if one expands assuming β^4 is small.

IV. LORENTZ INVARIANCE

The derived propagators violate Lorentz invariance. It is therefore worthwhile discussing in some detail the nature of the violation. There are two distinct origins for it, which we will discuss in the following two subsections.

A. Choices of polymerization

First of all, one has violation of Lorentz invariance due to the polymerization. This can be seen in terms like the dispersion relation implied by the denominator of (89),

$$\omega_1^2 - p(n_1)^2\left(1+\alpha_2\beta^4\right). \tag{90}$$

It should be noted that these terms depend on the value of the polymerization parameter β. The order in β at which these terms appear depends on choices made at the time of polymerization. To see this, let us write the polymerized momentum term in Hamiltonian as

$$\frac{c\sin(\beta P_\phi(i))}{\sqrt{2\epsilon}\beta} + \frac{(1-c)\sin(3\beta P_\phi(i)}{3\sqrt{2\epsilon}\beta} \tag{91}$$

and try to find c such that we are left only with the non-perturbative term in $P_\phi(i)$ and a perturbative term in β^4, thus neglecting the β^2 term. This way we can analyze just the effects of β^4 order term in the propagator. Expanding (91) in β we get

$$\frac{P_\phi(i)^2}{2\epsilon} + \left(\frac{4}{3}\frac{cP_\phi(i)^4}{\epsilon} - \frac{3}{2}\frac{P_\phi(i)^4}{\epsilon}\right)\beta^2 + \left(\frac{8}{9}\frac{c^2P_\phi(i)^6}{\epsilon} - \frac{8}{3}\frac{cP_\phi(i)^6}{\epsilon} + \frac{9}{5}\frac{P_\phi(i)^6}{\epsilon}\right)\beta^4. \tag{92}$$

Obviously from the coefficient of β^2 we see that setting $c=\frac{9}{8}$, the β^2 order term cancel and we are left with only the non-perturbative term and a perturbative term in β^4 which is

$$H_{\rm int}(i) = -\frac{3}{40\epsilon}P_\phi(i)^6\beta^4. \tag{93}$$

This would lead to corrections of order β^8 instead of β^4 in (89). We therefore see that the order in β at which corrections appear can be shifted arbitrarily by choosing suitable polymerizations of the theory and therefore, assuming that the polymerization parameter is small, one can make the corrections as small as desired.

It is interesting to emphasize that these corrections do not necessarily involve the Planck length. Unlike when one polymerizes the gravitational variables, there is no a priori reason to relate the polymerization parameter of a scalar field to the quantum of area. The reason for having a relation between the parameter and the area in the gravitational variables is because one is dealing with a true holonomy along a spatial loop which encloses an amount of area. In the case of the scalar field however, one is dealing with point holonomies and therefore they do not enclose area. These polymerization dependent Lorentz violations are also not of the form conjectured by Hořava in his "gravity at the Lifshitz point", [5] since there the corrections were Planck scale dependent. These corrections amount to a redefinition of the speed of light for a massless scalar field. This could lead to experimental problems if similar redefinitions do not occur for other fields, since differences in the speed of propagation of massless fields is severely constrained experimentally.

In addition to the Lorentz violations due to the polymerization there is the issue that we are working in a discrete theory for which we have failed to find a continuum limit. Since we have been unable to find a ground state for which the master constraint is zero, we worked with a variationally found state that minimized the master constraint. The minimum found was achieved with a finite lattice spacing. We found that the minimum of the master constraint occurs at a spacing large compared to the Planck length but small compared to particle physics scalings. This finiteness of the lattice implies that expressions like (89) are only approximate and there are corrections that go as the momentum to the fourth power times the lattice spacing squared. Those corrections are of Hořava type. This is good since Hořava has argued that such corrections help make theories finite, as one would expect in a lattice treatment like the one we pursue.

A last point here is that we have discussed the corrections due to polymerization and due to the lattice discreteness separately where in reality they are not separate. When we studied the corrections due to polymerization we took the continuum limit to get simple expressions. In reality, if one kept on working on the lattice till the end there would appear terms involving the lattice spacing as well in the corrections due to polymerization.

B. The arguments of Collins et al.

Collins et al. [6] have argued that Lorentz violations of the second kind considered in the previous section (more precisely, the corrections that depend on the Planck scale) could lead to unacceptably large effects when one considers interactions at one loop level. One has to be careful in applying their arguments directly to the propagators we discussed in the previous section since they were derived in the low energy limit, expanding in powers of the momentum. It is of course not legitimate to expand something in powers of a given parameter, keeping the lowest order, and then evaluating the expression for large values of the expansion parameter. So in order to reconstruct the argument of Collins et al. for our case, we would need the full expression of the propagators, which we do not have. One can sketch how it is likely to go. The terms we have neglected are due to the fact that the theory is put on a lattice. One knows that the lattice propagator for a scalar field takes the form,

$$\frac{1}{-m^2+\omega^2-\sum_i \frac{\sin^2(ap_i)}{a^2}}, \tag{94}$$

where a is the lattice spacing. The presence of the sine function implies that for large values of the momentum, the quantity remains finite. In particular, if one re-does the calculations of Collins et al. with such propagators, one finds that it leads to large Lorentz violations, due the asymmetric treatment of space and time. However, it would be hasty to conclude that this is a problem. The reason for this is that in general relativity one is not supposed to take the zeroth component of the coordinates as a time variable. What we have done in this paper is to work out a gauge fixed quantization of the scalar field, which implies a specific choice of coordinates classically. However, in a generally covariant theory one should really use physical systems of reference through the introduction of real "clocks and rods". This, in turn solves the "problem of time" of such theories. In a more realistic calculation than the one done here one would have other matter fields present that can be used as "clocks and rods" to measure time and space.

In generally covariant theories one should construct and ask physical questions about observables. This, in particular, applies to the calculation of Green's functions, as was already noted by DeWitt [12]. These would not correspont to the propagators we have considered up to now here, which are constructed in terms of the coordinates, but would have to be cast in terms of the times and distances measured by the physical "clocks and rods", so the resulting propagators are Dirac observables,

$$D\left(T_1,T_1,\vec{X}_1\vec{X}_2\right)=\int dt_1\int d^3x_1\int dt'_1\int d^3x'_2 D\left(t_1,t_2,\vec{x}_1\vec{x}_2\right)\mathcal{P}\left(t_1,T_1\right)\mathcal{P}\left(t_2,T_2\right)\mathcal{P}\left(\vec{x}_1,\vec{X}_1\right)\mathcal{P}\left(\vec{x}_2,\vec{X}_2\right), \tag{95}$$

and we are considering a situation where space is locally flat, otherwise the integrals should involve square roots of the determinant of the metric. The above expression can be derived in detail in the discussion of the problem of time starting from conditional probabilities of evolving Dirac observables, see [13, 14] for details. The quantities $\mathcal{P}$ are probability distributions that tell us what is the chance that given a value of a variable, say, t_1 the real clock is measuring T_1 and similarly for the others. Generically, these probability distributions will be highly peaked —provided one chose sensible clocks and rods—, indicating that there is little difference between the parameter time t_1 and the clock time T_1 and similarly for the spatial rods. The width of the distributions will depend on the physical details of the clocks and rods chosen, but the important point is that there exist fundamental physical limitations that dictate that the widths cannot be arbitrarily small [15]. These limitations in fact state that the widths should be considerably larger than the Planck scale (and of the lattice spacing considered in this paper). This introduces naturally a cutoff in four-momentum space that implies that the Lorentz violating contributions we encountered above will be suppressed.

For instance, let us just concentrate on the effect of the clock, as the one from the rods is similar. We assume that $\mathcal{P}(t,T)$ is an approximation of the Dirac delta given by a step function of width 2σ. In that case one can carry out the integral explicitly to get,

$$D\left(T,\vec{x},T',\vec{x}'\right)=\int_{-\pi/a}^{\pi/a} d^3p\frac{e^{i\vec{p}\cdot\vec{x}}}{2\omega_a}\frac{\sin^2\left(\omega_a\sigma\right)}{\omega_a^2\sigma^2}e^{-i\omega_a|T-T'|}, \tag{96}$$

where $\omega_a=\sqrt{m^2+\sum_j \frac{\sin^2(ap_j)}{a^2}}$. In usual quantum gravity scenarios, σ is proportional to some power of the Planck length and grows with time [14]. We notice that this introduces an ultraviolet cutoff in ω_a.

It is worthwhile discussing how does Lorentz invariance emerge in a context like this (a point emphasized by Rovelli and Speziale in [16]). When one introduces a set of clocks and rods $(T,\vec{X})$, one is manifestly breaking Lorentz invariance. The latter is recovered in the sense that if one takes the same set of clocks and rods as before (or one

similarly constructed) and boosts it $(T', \vec{X}')$ and one carries out the above calculation one will get that the physical propagator is invariant,

$$D\left(T_1, T_2, \vec{X}_1, \vec{X}_2\right) = D\left(T_1', T_2', \vec{X}_1', \vec{X}_2'\right). \tag{97}$$

In order for this equality to hold we need two conditions: that the uncertainties in the clocks and the rods be the same (otherwise that would automatically violate Lorentz invariance), and the second one that for small values of the momentum the propagators considered have the usual Lorentz invariant form.

V. DISCUSSION

We have studied the propagator in a polymerized scalar field theory with spherical symmetry. This requires defining a vacuum, which we took to be the Fock vacuum based on our experiences in Paper I. In this paper we further confirmed that this vacuum is adequate by computing the expectation value of the master constraint polymerized and expanded to leading order in the polymerization parameter and noting that the corrections introduced in the expectation value by the polymerization are very small. We then proceeded to study the propagator to leading order in the polymerization parameter for two different choices of polymerization: either polymerizing the field or its canonically conjugate momentum. We ended up with propagators that had Lorentz violations of two different types, one stemming from the polymerization of the scalar field and the other from the discreteness that is remnant from the uniform discretization procedure, since the state that minimizes the expectation value of the master constraint does so for a finite lattice spacing. This could be a temporary limitation until a better state is found, or it could well be that such a state actually does not exist.

The Lorentz violation due to polymerization can be made arbitrarily small by a suitable choice of the polymerization parameter. This is because in the case of a scalar field this parameter is not obviously associated with an area and therefore not limited by the minimum area eigenvalue as is the case for gravitational variables. The order of the violation in the parameter can also be changed by choices in polymerization. We also argued that the Lorentz violations that arise due to the use of a lattice are not of the type considered by Collins et al. and that if one uses real clocks and rods to characterize space-time points in such a way that propagators are Dirac observables, potential divergences in the integrals on the frequencies are contained and may not lead to large Lorentz violations either.

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Appendix

Since the construction carried out is rather elaborate, it is worthwhile spelling out for the readers the various assumptions made to reach the results of this paper.

a) We started in Paper I by attempting to find a “vacuum” for the combined gravity and scalar field system in spherical symmetry on a lattice. We polymerized the gravitational variables but as a first exploration we kept the scalar field not polymerized. By vacuum in this context we mean a state that minimizes the master constraint on the lattice with eigenvalue zero but at the same time that minimizes the energy of the matter Hamiltonian. We chose the simplest factor ordering of the master constraint which made it self-adjoint.

b) To find the vacuum we proceeded variationally, using a trial state. The trial state consisted of Gaussians at each lattice point centered around the classical solution (flat space with a deficit angle) for the gravitational variables. For the scalar field the trial state was a Fock state modified by the gravitational background and quantum corrections. The deficit angle arises due to the zero point energy of the scalar field, which in $1+1$ dimensions does not yield a local curvature (a cosmological constant as in 4 dimensions) and only produces global curvature. The total trial state was assumed to be the direct product of the Gaussians and the Fock state. For calculations involving the vacuum it is reasonable to assume no correlations. This assumption would be wrong for excited states.

c) We proceeded to minimize the master constraint given the variational trial state. The minimization parameters were the widths of the Gaussians for the gravitational variables. We found that the minimum was achieved with essentially constant values for the parameters along the lattice. The minimum value of the master constraint did not correspond to the lattice spacing going to zero. In fact the expectation value diverges in that limit. The theory does not have a continuum limit and the minimum of the master constraint is not zero, but is very small and so it is the ideal lattice spacing. It is large compared to the Planck length but still very small for particle physics standards. This concluded Paper I.

d) In this paper we wished to study the propagator of the polymerized scalar field. First we had to convince ourselves that the vacuum of Paper I was still useful, since it was derived without the assumption that the field was polymerized. We re-evaluated the expectation value of the master constraint with the scalar field polymerized on the vacuum of Paper I. We found that the value differed very little from the one found in Paper I. This validated the use of the vacuum of Paper I in the polymerized context.

e) We studied the polymerized scalar field treating the polymerization parameter as being small, thinking of the un-polymerized theory as a "free" theory and the extra terms stemming from the polymerization as perturbations. We then applied standard quantum field theory techniques to find the propagator.

f) However, we were on the lattice, so we used certain approximations to evaluate summations by taking the continuum limit and evaluating integrals. This way, to leading order in the lattice spacing the approximation is good.

g) We found that the propagators acquired Lorentz violating corrections due to the polymerization. Some of the terms not explicitly evaluated in the approximation f) also lead to Lorentz violations.

h) We discussed if the Lorentz violations could cause problems as those discussed by Collins et al. We argued that it may be possible that they do not, if one carries out a proper treatment of the problem of time and loop quantum gravity ends up providing a regularization of the matter fields with certain properties.

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